Chapter

2

Logarithms

2.1 Introduction to Logarithms

Imagine a world before calculators and computers when mathematicians had to do complex calculations involving multiplication and division of large numbers. It took tremendous time and effort often involving lengthy calculations.

Opening Puzzle: The Sound of Numbers

In a music studio, the sound engineer says:

“This speaker produces a sound that is 1000 times more intense than the softest sound we can hear.”

Instead of saying “1000 times,” scientists say:

Sound Level = $\log_{10} 1000 = 3$

Why 3?

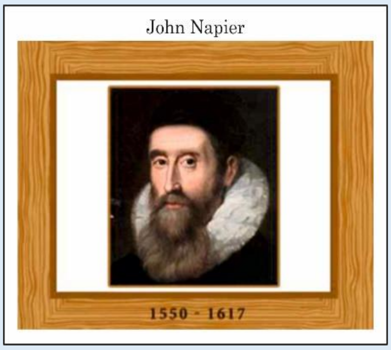
Because: $10^3 = 1000$

Then came **John Napier** with a revolutionary idea of effectively turning “multiplication into addition," and “division into subtraction” using logarithms. Logarithms is a tool that helps to do large calculations easily.

This approach saved scientists and fellow mathematicians a lot of tedious calculations.

Logarithms were introduced by John Napier (1550–1617), a Scottish mathematician. His method was different from the modern approach and was based on the relationship between arithmetic and geometric sequences. Later, Henry Briggs refined this idea and developed common logarithms (base 10), which made calculations easier.

Logarithm tables were widely used in science and engineering to simplify multiplication and division until electronic calculators became common. Even today, logarithms remain important in mathematics, especially natural logarithms (base *e*), which are widely used in calculus.



2.2 Understanding Logarithms as the Inverse of Exponents

We have earlier learnt about squares and cubes. For example:

- $10^2 = 100$ (10 squared is 100)
- $10^3 = 1000$ (10 cubed is 1000)

In these cases, we have taken the **base** 10, their **exponents** as 2 and 3. We find the value of base raised to the given exponent. But what if we know the resultant value and the base, and we want to find the exponent?

This power or exponent to which the base is raised is called a **Logarithm**.

- **Exponential Form:** $10^x = 100$
- **Logarithmic Form:** $\log_{10} 100 = x$

We already know about **exponents**: $2^3 = 8$

This means:

- Base = 2
- Exponent = 3
- Resultant = 8

Let us ask a different question: **2 raised to what power gives 8?**

That power is 3. So we say that logarithm of 8 to the base 2 is 3.

2.2.1 Understanding Logarithms through powers of 10

Let us look at powers of 10:

Powers of 10	Expressed in logarithmic form:
$10^0 = 1$ $10^1 = 10$ $10^2 = 100$ $10^3 = 1000$ $10^4 = 10000$	$\log_{10} 1 = 0$ $\log_{10} 10 = 1$ $\log_{10} 100 = 2$ $\log_{10} 1000 = 3$ $\log_{10} 10000 = 4$
$10^{-4} = 0.0001$ $10^{-3} = 0.001$ $10^{-2} = 0.01$ $10^{-1} = 0.1$ $10^0 = 1$	$\log_{10} 0.0001 = -4$ $\log_{10} 0.001 = -3$ $\log_{10} 0.01 = -2$ $\log_{10} 0.1 = -1$ $\log_{10} 1 = 0$

Definition: For any positive number b (where $b > 0$ and $b \neq 1$) and a positive number a if $b^x = a$ then $\log_b a = x$. We read this as " logarithm of a to the base b is x" .

For example, $32 = 2^5$ we can write $5 = \log_2 32$. These two statements are **equivalent** and we indicate this by using the symbol $\Leftrightarrow$. We write: $32 = 2^5 \Leftrightarrow \log_2 32 = 5$.

In general, $b^x = a$ where $a, b > 0$ and $b \neq 1$ and $\log_b a = x$ are **equivalent** statements and we write

$$b^x = a \quad \Leftrightarrow \quad \log_b a = x$$

Example 1: Write an equivalent form:

(a) Logarithmic form for $9^{\frac{1}{2}} = 3$

(b) Exponential form of $\log_5 625 = 4$

Math Talk:
Can we find logarithm
of a negative number?
Why or why not?

Solution:

(a) $9^{\frac{1}{2}} = 3 \quad \Leftrightarrow \quad \log_9 3 = \frac{1}{2}$

(b) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

Example 2: Rewrite $2^0 = 1$ in logarithmic form:

Solution: $2^0 = 1 \quad \Leftrightarrow \quad \log_2 1 = 0$

Math Talk: Is this true
for all bases?

EXERCISE 2.1

1. Write an equivalent logarithmic statement for:

   (a) $5^3 = 125$ (b) $(2)^5 = 32$

   (c) $(7)^{-1} = \frac{1}{7}$ (d) $(3)^{\frac{-1}{2}} = \frac{1}{\sqrt{3}}$

2. Write an equivalent exponential statement for:

   (a) $\log_2 16 = 4$ (b) $\log_9 81 = 2$

   (c) $\log_5 \sqrt{5} = \frac{1}{2}$ (d) $\log_2\left(\frac{1}{2}\right) = -1$

3. Find the value of

   (a) $\log_{10} 1000$ (b) $\log_6 36$ (c) $\log_2 64$

2.3 Logarithms Properties

The definition and the assumptions of logarithm are as follows:

Definition of Logarithm:

For $a > 0$, $a \neq 1$ and $x > 0$,

If $y = \log_a x$, then $a^y = x$.

Equivalently, $\log_a x = y \Leftrightarrow a^y = x$.

Assumptions:

- $a > 0$, $a \neq 1$ (The Base)
- $M > 0$, $N > 0$ (Positive real number)

These conditions ensure all logarithms in the following proofs, are defined.

Properties of Logarithms

Product Rule

Statement: $\log_a(MN) = \log_a M + \log_a N$

Proof: Let $x = \log_a(MN)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = MN$, $a^y = M$ and $a^z = N$

Therefore, $a^{y+z} = a^y a^z = MN \Rightarrow y + z = \log_a(MN)$

Thus, $\log_a M + \log_a N = \log_a(MN)$

Quotient Rule

Statement: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

Proof: Let $x = \log_a\left(\frac{M}{N}\right)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = \left(\frac{M}{N}\right)$, $a^y = M$ and $a^z = N$

Therefore, $a^{y-z} = \frac{a^y}{a^z} = \left(\frac{M}{N}\right) \Rightarrow y - z = \log_a\left(\frac{M}{N}\right)$

Thus, $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$

Power Rule

Statement: $\log_a\left(M^k\right) = k\log_a M$

Proof: Let $x = \log_a\left(M^k\right)$ and $y = \log_a M$

Then, $a^x = M^k$, $a^y = M$

Therefore, $a^x = a^{ky} \Rightarrow x = ky$

Thus, $\log_a\left(M^k\right) = k\log_a M$

Change of Base Formula

Statement: $\log_a(M) = \dfrac{\log_b M}{\log_b(a)}$, For $b > 0$, $b \neq 1$

Proof: Let $x = \log_a(M) \Rightarrow a^x = M$

Now take log with base b on both the sides,

$\log_b(a^x) = \log_b M \Rightarrow x\log_b a = \log_b M$

Thus, $x = \log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$

Log of 1

Statement: $\log_a(1) = 0$

Proof: Let $x = \log_a 1 \Rightarrow a^x = 1$

Since, $a^0 = 1$, for any $a > 0, a \neq 1$

$a^0 = 1 = a^x \Rightarrow x = 0$

Thus, $x = \log_a(1) = 0$

Log of *a* number to the same base

Statement: $\log_a a = 1$

Proof: Let $x = \log_a a \Rightarrow a^x = a$

Since, $a^1 = a$, for any $a > 0, a \neq 1$

$a^1 = a^x \Rightarrow x = 1$

Thus, $x = \log_a a = 1$

Let us summarize the properties of logarithms we proved above in the table below:

For any base a, $a > 0$, $a \neq 1$ and M, N > 0

Property Name	Logarithmic Notation	What it does
Product Rule	$\log_a(M \times N) = \log_a M + \log_a N$	Turns multiplication into addition
Quotient Rule	$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$	Turns division into subtraction
Power Rule	$\log_a\left(M^k\right) = k \times \log_a M$	The exponent comes down
log of number to the same base	$\log_a(a) = 1$	Log of any number to the same base is 1
log of 1	$\log_a(1) = 0$	Logarithm of 1 to any base is zero
Base changing property	$\log_a n = \frac{\log_b n}{\log_b a}$	Base a is changed to any other base b ($b > 0$ and $b \neq 1$)

Remember: For positive values of m, n, x, y and $a > 0$, $a \neq 1$,

- $\log_a(m+n) \neq \log_a m + \log_a n$
- $\log_a(m-n) \neq \log_a m - \log_a n$
- If $x \neq y$ then $\log_a x \neq \log_a y$

Example 3: Write it as a single logarithm

(a) $\log_7 3 + \log_7 5$

(b) $\log_2 9 - \log_2 3$

(c) $\log_4 3 + \log_4 6 - 3\log_4 2$

(d) $1 + \log_3 5$

Solution:

(a)
$$\begin{aligned} \log_7 3 + \log_7 5 &= \log_7(3 \times 5) \\ &= \log_7 15 \end{aligned}$$

(b)
$$\begin{aligned} \log_2 9 - \log_2 3 &= \log_2\left(\frac{9}{3}\right) \\ &= \log_2 3 \end{aligned}$$